

Survival of branching processes in random environments

E. Dyakonova*, V. Vatutin† and S. Sagitov ‡

October 28, 2011

Abstract

This review paper presents the known results on the asymptotics of the survival probability and limit theorems conditioned on survival of critical and subcritical branching processes in IID random environments. The key assumptions of the family of population models in question are: non-overlapping generations, independent reproduction of particles within a generation, independent reproduction laws between generations. This is a biologically important generalization of the time inhomogeneous branching processes. The assumption of IID (independent and identically distributed) random environments reflects uncertainty in the future (as well as historical) reproduction regimes in actual populations.

This review focusses on a particular range of questions of prime interest for the authors. The reader should be aware of the fact that there are many very interesting papers covering other issues on branching processes in varying and random environments which are not mentioned here.

1 Introduction

There has been a constantly growing interest in branching processes in random environment (BPRES) following the pioneering publications by Smith and Wilkinson [47] in 1969 and Athreya and Karlin [13], [14] in 1971 (see [66] for a list of references up to 1985 and [1]–[12], [18]–[21], [26]–[30], [34]–[37], [43], [44], [48], and [52]–[64] for some more recent results). A BPRES is a stochastic population model where individuals constituting a generation reproduce independently according to a common offspring distribution. The random environment feature of the model means that the reproduction laws randomly change from one generation to the other. In this review we summarize some recent findings on

*Steklov Mathematical Institute, Gubkina str. 8, 119991, Moscow, Russia. Email address: elena@mi.ras.ru. Supported in part by the RFBR, Grant 11-01-00139.

†Steklov Mathematical Institute, Gubkina str. 8, 119991, Moscow, Russia. Email address: vatutin@mi.ras.ru. Supported in part by the RFBR, Grant 11-01-00139.

‡Chalmers University and University of Gothenburg, 412 96 Gothenburg, Sweden. Email address: serik@chalmers.se. Supported by the Swedish Research Council grant 621-2010-5623.

critical and subcritical discrete time BPRES under a key assumption that the reproduction laws of particles of different generations are independent and identically distributed (IID). The results reported here are mainly due to the joined efforts of the Russian-German team of researchers consisting of V. Afanasyev, C. Boinghoff, E. Dyakonova, J. Geiger, K. Fleischmann, G. Kersting, V. Vatutin, V. Wachtel and others (see also [18] for an earlier review).

In the study of stochastic processes in random environment it is crucial to recognize the difference between quenched and annealed approaches. Under the *quenched approach* characteristics of a BPRES such as the survival probability at time n are treated as random variables, where the source of randomness is due to uncertainty in possible realizations of the environment (see, for example, [13], [14], [15], [49], [55] and the relevant literature in [66]). While the *annealed approach* studies the mean values of the mentioned characteristics as a result of averaging over possible realizations of the environment (see [2], [3], [6], [19], [34], [42], [43], [52], [53] and the bibliography in [66]). The annealed approach can be viewed as a summary of the more detailed quenched analysis. One has to be aware that in biological applications the predictions by the annealed approach might be misleading: in reality the environmental history, being uncertain, is unique.

The remainder of the paper is organized as follows. Section 2 introduces an important classification of BPRES into three major classes: supercritical, critical, and subcritical branching processes. Here we discuss the key assumptions imposed on the reproduction laws in the critical and subcritical cases. Section 3 contains asymptotic results obtained using the annealed approach. In Section 4 we summarize the results for the multi-type BPRES where individuals within the same generation are allowed to have different reproduction laws. The quenched results for the critical BPRES are presented in Section 5. The last Section 6 is devoted to the reduced critical BPRES.

Observe that in the setting of random environment we use different fonts for the probability and expectation operators depending on whether we condition on the environment $(\mathcal{P}, \mathcal{E})$ or not (\mathbb{P}, \mathbb{E}) . Yet another font (\mathbf{P}, \mathbf{E}) is used when we introduce an auxiliary probability measure in the subcritical case (compare for example Conditions 5 and 7). Constants denoted by c may take different values in different formulae (the same reservation holds for c_1, c_2).

2 Classification of BPRES

The basic classification of the branching processes in a constant environment recognizes three major reproduction regimes: subcritical, critical and supercritical. It is based on the mean value m of the offspring number and these three classes are defined by the following relations: $m < 1$, $m = 1$, and $m > 1$. Adding the feature of random environment makes the classification issue less straightforward due to random fluctuations of the consecutive offspring mean values.

2.1 BPREs and associated random walks

A branching process in a *varying* environment is most conveniently described in terms of a sequence of probability generating functions (PGFs)

$$f_n(s) = \sum_{i=0}^{\infty} q_i^{(n)} s^i,$$

where $q_i^{(n)}$ stands for the probability that a particle from generation $n-1$ contributes to the next generation n by producing exactly i offspring (independently of other particles). Time inhomogeneous Markov chain $\{Z_n\}$ describing the fluctuations in population sizes is then characterized by the iterations of the consecutive reproduction PGFs

$$\mathbb{E}[s^{Z_n}] = f_1(f_2(\dots(f_n(s))\dots)), \quad (1)$$

if we assume that $Z_0 = 1$. Taking derivatives in the previous relation gives an expression for the expected population size

$$\mathbb{E}[Z_n] = m_1 \dots m_n, \quad (2)$$

where $m_n = f'_n(1)$ stands for the mean offspring number.

An important example is the linear-fractional case

$$f_n(s) = r_n + (1 - r_n) \frac{t_n s}{1 - (1 - t_n)s}$$

fully characterized by a sequence of pairs of parameters $(r_n, t_n) \in [0, 1) \times (0, 1]$. The iteration of linear-fractional PGFs is again linear-fractional

$$f_1(f_2(\dots(f_n(s))\dots)) = r^{(n)} + (1 - r^{(n)}) \frac{t^{(n)} s}{1 - (1 - t^{(n)})s}$$

with

$$\begin{aligned} \frac{1 - r^{(n)}}{t^{(n)}} &= m_1 \dots m_n, \\ \frac{1 - t^{(n)}}{t^{(n)}} &= \frac{1 - t_n}{t_n} + \frac{1 - t_{n-1}}{t_{n-1}} m_n + \dots + \frac{1 - t_1}{t_1} m_2 \dots m_n. \end{aligned}$$

It follows,

$$\frac{1}{t^{(n)}} = 1 + \frac{1 - t_n}{1 - r_n} m_n + \frac{1 - t_{n-1}}{1 - r_{n-1}} m_{n-1} m_n + \dots + \frac{1 - t_1}{1 - r_1} m_1 \dots m_n,$$

and since $\mathbb{P}(Z_n = 0) = r^{(n)}$,

$$\frac{1}{\mathbb{P}(Z_n > 0)} = e^{-s_n} + \sum_{k=1}^n a_k e^{-s_{k-1}},$$

with $a_k = \frac{1-t_k}{1-r_k}$ and $s_k = \ln m_1 + \dots + \ln m_k$. The obtained equality indicates that the asymptotics of the survival probability $\mathbb{P}(Z_n > 0)$ under certain assumptions is governed by the minimal value of consecutive expectations of population sizes

$$\min(e^{s_1}, \dots, e^{s_n}) = e^{\min(s_1, \dots, s_n)}.$$

Notice also that in the linear-fractional case $(Z_n - 1 \mid Z_n > 0)$ is geometrically distributed $\text{Geom}(t^{(n)})$.

Given a sequence of *random* PGFs (F_1, F_2, \dots) one can speak of a BPPE $\{Z_n\}_{n=0}^\infty$. Throughout we will assume that

- the defining random PGFs in are IID so that all $F_n \stackrel{d}{=} F$ have the same marginal distribution,
- the BPPE starts by a single particle, $Z_0 = 1$, unless it is clearly stated otherwise.

For BPPEs the relation (1) transforms into

$$\mathbb{E}[s^{Z_n} \mid F_1 = f_1, F_2 = f_2, \dots] = f_1(f_2(\dots(f_n(s))\dots)).$$

Throughout we will denote the probability and expectation conditioned on the environment by

$$\mathcal{P}(\cdot) := \mathbb{P}(\cdot \mid F_1, F_2, \dots), \quad \mathcal{E}[\cdot] := \mathbb{E}[\cdot \mid F_1, F_2, \dots].$$

Observe that the quenched counterpart of the relation (2) can be viewed as a realization of a Markov chain $\mathcal{E}[Z_n] = e^{S_n}$ defined by the so called *associated random walk* (ARW):

$$S_0 = 0, \quad S_n = X_1 + \dots + X_n, \quad \text{where } X_n = \ln F'_n(1).$$

Here the jump sizes are IID random variables with

$$X_n \stackrel{d}{=} X := \ln F'(1) = \ln \mathcal{E}[\xi],$$

where ξ is a random variable representing the offspring number for a single particle reproducing in a random environment.

The key representation $\mathcal{E}[Z_n] = e^{S_n}$ leads to the following extended classification of branching processes: a BPPE is called (A) supercritical, (B) subcritical, (C) non-degenerate critical and (C₀) degenerate critical, if its ARW a.s. satisfies one of the following conditions

- (A) $\lim_{n \rightarrow \infty} S_n = +\infty$,
- (B) $\lim_{n \rightarrow \infty} S_n = -\infty$,
- (C) $\limsup_{n \rightarrow \infty} S_n = +\infty$ and $\liminf_{n \rightarrow \infty} S_n = -\infty$,
- (C₀) $S_n = 0$ for all n .

This classification, due to [8], is based on the crucial fact (see for example, [31], ch.12, §2) that *any* IID random walk can be attributed to one of these

four classes. For the processes in which $\mathbb{E}[X]$ exists this classification coincides with the standard classification of branching processes: a BPRE is called supercritical, critical or subcritical if $\mathbb{E}[X] > 0$, $= 0$, or < 0 , respectively (see, for instance, [15]). However, the extended classification seems to be more natural and is justified by a series of results published in [8], [30], [55], [56], [57], [58]. In particular, in view of the estimate

$$\mathcal{P}(Z_n > 0) = \min_{1 \leq k \leq n} \mathcal{P}(Z_k > 0) \leq \min_{1 \leq k \leq n} \mathcal{E}[Z_k] = e^{\min(S_1, \dots, S_n)}$$

it follows that the probability of extinction of subcritical and non-degenerate critical BPRE equals 1 a.s. Moreover, it was shown in [8], [30], [55], [56], [57], and [58] that the asymptotic properties of the non-degenerate critical BPRE with $\mathbb{E}[X] = 0$ are quite similar to those in which $\mathbb{E}[X]$ does not exist.

In what follows we mostly focus on classes (B) and (C) excluding the degenerate critical class (C_0) . Therefore, in the case (C) we will speak of critical BPREs omitting the specification 'non-degenerate'. Note that by excluding the class (C_0) we disregard the critical Galton-Watson processes in constant environments.

2.2 The Spitzer-Doney condition in the critical case

We further specify the general condition (C) by assuming that the ARW satisfies the classical *Spitzer condition* with parameter ρ :

$$\frac{1}{n} \sum_{k=1}^n \mathbb{P}(S_k > 0) \rightarrow \rho \in (0, 1) \quad \text{as } n \rightarrow \infty.$$

According to [23] the Spitzer condition is equivalent to the following *Doney condition* with parameter ρ

$$\mathbb{P}(S_n > 0) \rightarrow \rho \in (0, 1) \quad \text{as } n \rightarrow \infty.$$

In this text we refer to this condition as the *Doney-Spitzer condition* expressing it in the form

$$\rho = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbb{P}(S_k > 0) = \lim_{n \rightarrow \infty} \mathbb{P}(S_n > 0) \in (0, 1). \quad (3)$$

It is well-known that any random walk satisfying (3) is of the oscillating type (see, e.g., Section XII.7 in [31]). We note that condition (3) covers random walks satisfying Condition 1 (see below) as well as all non-degenerate symmetric random walks. In these cases $\rho = 1/2$.

Condition 1 *The distribution of X has zero mean and variance $0 < \sigma^2 < \infty$. It is non-lattice.*

Of course, one can consider the lattice case as well. However, this leads to unnecessary complications in the statements of the respective results which we prefer to avoid.

Other examples where the Doney-Spitzer condition fulfills are provided by random walks in the domain of attraction of some stable law, see Condition 2 below. We will often refer to this weaker version of (3) as it allows to relax some extra conditions imposed on the random environment.

Condition 2 *The distribution of X is non-lattice and belongs without centering to the domain of attraction of a stable law with index $\alpha \in (0, 2]$ and skewness parameter ψ , such that*

- $|\psi| < 1$, if $0 < \alpha < 1$,
- $|\psi| \leq 1$, if $1 < \alpha < 2$,
- $\psi = 0$ if $\alpha = 1$ or $\alpha = 2$.

Recall that the condition that the centering constants are zero is redundant, if $0 < \alpha < 1$, and that a stable law from Condition 2 has the characteristic function (see, e.g., Theorem 8.3.2 in [17])

$$\chi_{\alpha, \psi}(t) = \exp \left(-c|t|^\alpha \left(1 - i\psi \frac{t}{|t|} \tan \frac{\pi\alpha}{2} \right) \right), \quad t \in \mathbb{R},$$

where $c > 0$ is a scaling parameter. This stable law has finite absolute moments of all orders $r < \alpha$. One can check (see, e.g., Section 8.9.2 in [17]) that Condition 2 implies (3) with parameter

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan \left(\psi \tan \frac{\pi\alpha}{2} \right) \in (0, 1). \quad (4)$$

Let

$$L_n := \min(S_1, \dots, S_n), \quad M_n := \max(S_1, \dots, S_n),$$

and

$$\tau(n) := \min\{i \in [0, n] : S_i = \min(0, L_n)\}$$

be the left-most point at which the ARW attains its minimal value on the time-interval $[0, n]$. It is known (see [46, Ch. IV, § 20]) that under the Doney-Spitzer condition

$$n^{-1}\tau(n) \xrightarrow{d} \tau, \quad n \rightarrow \infty, \quad (5)$$

where τ is a random variable having a Beta-distribution with parameters $(1 - \rho, \rho)$ and the symbol \xrightarrow{d} stands for the convergence in distribution.

Let

$$D := \sum_{k=1}^{\infty} k^{-1} \mathbb{P}(S_k = 0),$$

and

$$\begin{aligned}\gamma_0 &:= 0, & \gamma_{j+1} &:= \min(n > \gamma_j : S_n < S_{\gamma_j}), \\ \Gamma_0 &:= 0, & \Gamma_{j+1} &:= \min(n > \Gamma_j : S_n > S_{\Gamma_j}), \quad j \geq 0,\end{aligned}$$

be the strict descending and strict ascending ladder epochs of the ARW. Next we introduce two renewal functions

$$U(x) = \begin{cases} 1_{\{0 < x\}} + \sum_{j=1}^{\infty} \frac{\mathbf{P}(S_{\Gamma_j} < x)}{e^{-D}} & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases} \quad (6)$$

and

$$V(x) = \begin{cases} \sum_{j=0}^{\infty} \mathbf{P}(S_{\gamma_j} \geq -x) & \text{if } x > 0, \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (7)$$

It is known (see [57], Lemma 1) that (3) implies

$$\begin{aligned}\mathbb{E}[U(-X)1_{\{X < 0\}}] &= e^{-D}, \\ \mathbb{E}[U(x-X)1_{\{X < x\}}] &= U(x), \quad x > 0, \\ \mathbb{E}[V(x+X)] &= V(x), \quad x \geq 0.\end{aligned} \quad (8)$$

2.3 Refined classification in the subcritical case

According to (B) the ARW in the subcritical case should have a clear trend towards minus infinity. It turns out (see [1], [21], and [36]) that the behavior of a subcritical BPRE to a large extent is determined by the speed of the negative drift quantified by the parameter β , when exists, such that

$$\mathbb{E}[Xe^{\beta X}] = 0. \quad (9)$$

We will distinguish among three different sub-cases:

- *weakly* subcritical, if (9) holds with some $0 < \beta < 1$, which implies $\mathbb{E}[Xe^X] > 0$,
- *intermediately* subcritical, if (9) holds with $\beta = 1$, so that $\mathbb{E}[Xe^X] = 0$,
- *strongly* subcritical, if $\mathbb{E}[Xe^X] < 0$.

Notice that due to monotonicity of the function $\mathbb{E}[Xe^{\beta X}]$ in all three sub-cases we have $\mathbb{E}[X] < 0$. Clearly, the smaller is the positive value of β the closer we get to the critical case with $\mathbb{E}[X] = 0$. On the other hand, the subcritical case in a constant environment corresponds to X being a negative constant implying the strong subcriticality.

Observe next that $\mathbb{E}[e^{\beta S_n}] = \gamma^n$, where

$$\gamma = \mathbb{E}[e^{\beta X}]. \quad (10)$$

Let us introduce the following auxiliary measure \mathbf{P} with expectation \mathbf{E} . For any $n \in \mathbf{N}$ and any measurable, bounded function $\varphi : \Delta^n \times \mathbf{N}_0^{n+1} \rightarrow \mathbb{R}$, the measure \mathbf{P} is given by

$$\mathbf{E}[\varphi(F_1, \dots, F_n, Z_0, \dots, Z_n)] := \gamma^{-n} \mathbb{E}[\varphi(F_1, \dots, F_n, Z_0, \dots, Z_n) e^{\beta S_n}].$$

Notice that $\mathbf{E}[X] = 0$ making S a recurrent random walk under \mathbf{P} . Clearly, in the critical case the measures \mathbf{P} and \mathbb{P} coincide since $\beta = 0$.

Condition 3 *The distribution of X with respect to \mathbf{P} is non-lattice, has zero mean and belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2]$.*

Under Condition 3 there exists an increasing sequence of positive numbers a_n regularly varying at infinity

$$a_n = n^{1/\alpha} l_n \tag{11}$$

such that the scaled ARW $a_n^{-1} S_{nt}$ \mathbf{P} -weakly converges to a strictly stable Lévy-process L_t with parameter α . Here and elsewhere in the expressions like S_{nt} the index nt is understood as its integer part.

3 Annealed approach

Different trajectories of the ARW represent different scenarios of environmental history. The more favorable scenarios translate into higher ARW trajectories. If an ARW trajectory stays below zero for a longer period of time, the BPPE is doomed to die out. The annealed analysis of the survival of BPPE gives a summary picture of the system behavior after averaging over all successful ARW trajectories.

3.1 Asymptotic behavior of the survival probability

The first result on the asymptotic behavior of the survival probability for the critical BPPEs has appeared in the seminal paper by Kozlov [42]. Kozlov has shown for BPPEs satisfying Condition 1 that under certain additional assumptions

$$c_1 n^{-1/2} \leq \mathbb{P}(Z_n > 0) \leq c_2 n^{-1/2},$$

where $0 < c_1 \leq c_2 < \infty$. Additionally, he proved that in the linear-fractional case

$$\mathbb{P}(Z_n > 0) \sim c n^{-1/2}, \quad c > 0, \quad n \rightarrow \infty. \tag{12}$$

These results were published in 1976. Only in 2000 Geiger and Kersting [35] were able to demonstrate the validity of (12) for arbitrary BPPEs satisfying Condition 1. To establish this result Geiger and Kersting developed a new powerful method of proving conditional limit theorems for BPPEs based on a

change of measures. This method, along with the idea of splitting trajectories of the ARW at the point of global minimum on the interval $[0, n]$, first suggested by Vatutin and Dyakonova [56], became one of the main tools for obtaining conditional limit theorems not only for the critical BPRES but for the subcritical BPRES as well.

3.1.1 Critical case

The main idea in finding the asymptotic behavior of the survival probability of critical BPRES under annealed approach is to show that

$$\mathbb{P}(Z_n > 0) \sim \theta \mathbb{P}(\min(S_1, \dots, S_n) \geq 0), \quad n \rightarrow \infty, \quad (13)$$

where $\theta > 0$. This is done by splitting trajectories of the ARW at the point of global minimum on the interval $[0, n]$. For the annealed approach the trajectory splitting method was first used by Dyakonova, Geiger, and Vatutin [30] and then further developed by Afanasyev, Geiger, Kersting, and Vatutin [8].

Recall that $X = \ln \mathcal{E}[\xi]$ is the logarithm of the mean for the conditional on the environment offspring number distribution

$$Q_i = \mathcal{P}(\xi = i), \quad i = 0, 1, 2, \dots \quad (14)$$

In terms of the conditional probabilities Q_i the random PGF F can be expressed as

$$F(s) = \mathcal{E}[s^\xi] = \sum_{i=0}^{\infty} Q_i s^i.$$

In [8] a higher moment assumption on the environment is given in terms of the standardized truncated second moment of the offspring number ξ :

$$\zeta(a) := \frac{\mathcal{E}[\xi^2 1_{\{\xi \geq a\}}]}{(\mathcal{E}[\xi])^2}. \quad (15)$$

Condition 4 For some positive ε and a

$$\mathbb{E}[(\ln^+ \zeta(a))^{\frac{1}{\varepsilon} + \epsilon}] < \infty \quad \text{and} \quad \mathbb{E}[V(X)(\ln^+ \zeta(a))^{1+\epsilon}] < \infty,$$

where $\ln^+ x := \ln(\max\{x, 1\})$.

Next we present a number of cases when this assumption is fulfilled.

1. If the offspring number has a bounded support, i.e.

$$\mathbb{P}(\xi \leq a^*) = 1 \quad (16)$$

for some a^* , then $\zeta(a) = 0$ \mathbb{P} -a.s. for all $a > a^*$. Obviously, in this case Condition 4 is valid. As a particular example we mention here the binary splitting reproduction law characterized by

$$\mathcal{P}(\xi = 2) = 1 - \mathcal{P}(\xi = 0) = e^X/2.$$

2. In view of relation (8), we have $\mathbb{E}[V(X)] = V(0) = 1$. Therefore, Condition 4 is satisfied, if $\zeta(a)$ is a.s. bounded from above for some $a \in \mathbf{N}_0$.

Observe that $\zeta(2) \leq 2\eta$, where

$$\eta := \frac{F''(1)}{(F'(1))^2} = \frac{\mathcal{E}[\xi(\xi-1)]}{(\mathcal{E}[\xi])^2} \quad (17)$$

implying that $\zeta(2)$ is bounded from above if η is bounded. This is the case when (14) is either a Poisson distribution with a random mean (when $\eta = 1$ a.s.) or a geometric distribution with a random mean (when $\eta = 2$ a.s.).

3. The renewal function $V(x)$ always satisfies $V(x) = O(x)$ as $x \rightarrow \infty$ and $V(x) = 0$ for $x < 0$. Therefore, as follows from Hölder's inequality, Condition 4 would be satisfied, provided

$$\mathbb{E}[(X^+)^p] < \infty \quad \text{and} \quad \mathbb{E}[(\ln^+ \zeta(a))^q] < \infty$$

for some $p > 1$ and $q > \max(\rho^{-1}, p(p-1)^{-1})$.

Under Condition 2 we have $\rho \leq \alpha^{-1}$ due to (4). This observation leads to the following weaker version of Condition 4.

Condition 5 *For some positive ε and a*

$$\mathbb{E}[(\ln^+ \zeta(a))^{\alpha+\varepsilon}] < \infty.$$

It was proved in [8] that if (3) holds together with Condition 4 or if Condition 2 with $|\psi| < 1$ and Condition 5 are valid, then there exists a positive finite number θ such that, as $n \rightarrow \infty$,

$$\mathbb{P}(Z_n > 0) \sim \theta \mathbb{P}(\min(S_1, \dots, S_n) \geq 0) \sim \theta n^{-(1-\rho)} l_n, \quad (18)$$

where l_1, l_2, \dots is a sequence varying slowly at infinity (consult [8] for representations of θ and the sequence l). Moreover, it was shown in [8] that for a trajectory of the ARW to give an essential contribution to the annealed probability of survival during the time interval $[0, n]$ this trajectory should pass the point of its global minimum over this time interval very soon after its start at time 0.

The result (18) gives the asymptotics of the tail distribution for the extinction time $T = \min\{k : Z_k = 0\}$ in view of the equality

$$\mathbb{P}(Z_n > 0) = \mathbb{P}(T > n).$$

The study of the asymptotic behavior of $\mathbb{P}(T = n)$ for the critical BPFE was initiated by Vatutin and Dyakonova [53]. They demonstrated that for the linear-fractional BPFEs under (3), Condition 1 and some additional conditions one has

$$\mathbb{P}(T = n) \sim cn^{-3/2}, \quad n \rightarrow \infty, \quad (19)$$

where $c \in (0, \infty)$. Afterwards, Böinghoff, Dyakonova, Kersting, and Vatutin [20] have proved (19) for the linear-fractional BPFEs under weaker conditions: (3), Condition 1, Condition 5 with $\alpha = 2$ plus the following Condition 6.

Condition 6 *There exists a constant $\chi \in (0, 1/2)$ such that*

$$\mathbb{P}(\chi \leq Q_0 \leq 1 - \chi, \quad \eta \geq \chi) = 1.$$

Vatutin and Wachtel [64] for the geometric case with

$$F_n(s) = \frac{e^{-X_n}}{1 + e^{-X_n} - s}, \quad n = 1, 2, \dots,$$

under Condition 2 restricted to $\alpha < 2$ and $|\psi| < 1$ have shown that similarly to (13) one has

$$\mathbb{P}(T = n) \sim \Theta \mathbb{P}(T^- = n) \quad \text{as } n \rightarrow \infty, \quad (20)$$

where $\Theta > 0$ and

$$T^- = \min\{k \geq 1 : S_k < 0\}.$$

3.1.2 Subcritical case

Weakly subcritical case. The best annealed results up to this moment for weakly subcritical BPRE were obtained by Afanasyev, Boinghoff, Kersting, and Vatutin [11]. It was proved that under Conditions 3 and 7 (see below) there exist numbers $0 < \kappa, \kappa' < \infty$ such that

$$\mathbb{P}(Z_n > 0) \sim \kappa \mathbb{P}(\min(S_1, \dots, S_n) \geq 0) \sim \kappa' \frac{\gamma^n}{na_n},$$

where γ is from (10) and a_n are from (11). Observe that this result is similar to that in the critical case (18), whereas it is no longer true in the intermediately and strongly subcritical cases (see e.g., [36]).

Condition 7 *For some positive ε and a*

$$\mathbf{E}[(\ln^+ \zeta(a))^{\alpha+\varepsilon}] < \infty.$$

Intermediately subcritical case. Afanasyev, Böinghoff, Kersting, and Vatutin [10] proved that given $\beta = 1$ and Conditions 3, 7 there are a constant $0 < \theta < \infty$ and a sequence l_n slowly varying at infinity such that

$$\mathbb{P}(Z_n > 0) \sim \theta \gamma^n \mathbb{P}(S_n < \min(S_1, \dots, S_{n-1})) \sim \frac{\gamma^n l_n}{n^{1-1/\alpha}}, \quad n \rightarrow \infty.$$

Under a slightly different assumption the last relation was first proved in [54].

Strongly subcritical case. Guivarc'h and Liu in [37] have shown for strongly subcritical processes satisfying

$$\mathbb{E}[\xi \ln^+ \xi] < \infty \quad (21)$$

that for some $0 < c \leq 1$

$$\mathbb{P}(Z_n > 0) \sim c (\mathbb{E}[\xi])^n, \quad n \rightarrow \infty. \quad (22)$$

This asymptotic formula was originally established by D’Souza and Hambly in [48] under an extra moment assumption (also a similar statement was proved in [48] assuming ergodicity of the environment). For the fractional-linear case the statement (22) was obtained in [4].

Condition (21) is a counterpart of the classical $x \log x$ condition for supercritical and subcritical branching processes in constant environment (see [15]). It holds, in particular, under the restriction (16).

3.2 Conditional limit theorems

Given that a BPPE has survived during a long period of time, what can be said about its demographic history? The theorems of this section reveal different successful survival strategies adjusted to the variable reproduction strength of the population living in a random environment.

3.2.1 Critical case

Afanasyev, Geiger, Kersting, and Vatutin [8] proved so far the best annealed conditional limit theorem. According to this theorem a surviving critical BPPE behaves in a ‘supercritical’ manner. Supercritical branching processes (whether classical or in random environment) grow exponentially fast $Z_n e^{-S_n} \rightarrow W$ a.s., where W is a non-degenerate random variable (see [15], [45]).

For integers $0 \leq r \leq n$ let $X_t^{r,n}$, $t \in [0, 1]$, be the rescaled generation size process given by

$$X_t^{r,n} = Z_{r+(n-r)t} \cdot e^{-S_{r+(n-r)t}}, \quad t \in [0, 1]. \quad (23)$$

In [8] it was proved that if (3) and Condition 4 (or Condition 2 with $|\psi| < 1$ and Condition 5) are valid, then,

$$\mathcal{L}(X_t^{r,n}, t \in [0, 1] \mid Z_n > 0) \xrightarrow{\mathcal{D}} \mathcal{L}(W_t, t \in [0, 1]) \quad \text{as } n \rightarrow \infty, \quad (24)$$

where the symbol $\xrightarrow{\mathcal{D}}$ stands for the convergence in distribution in the Skorokhod topology in space $D[0, 1]$ of càdlàg functions on the unit interval or in other spaces which we will meet later on. Here r_1, r_2, \dots is a sequence of positive integers such that $r_n \leq n$ and $r_n \rightarrow \infty$. This sequence is introduced to exclude the early part of the population history.

The limiting process $\{W_t, t \in [0, 1]\}$ is a stochastic process with a.s. constant paths, i.e., $\mathbb{P}(W_t = W \text{ for all } t \in [0, 1]) = 1$ for some positive random variable W . We see that the growth of the BPPE is mainly determined by the ARW, namely by the sequence $(e^{S_n})_{n \geq 0}$, with the fine structure of the random environment being summarized by W . Afanasyev [6] in the case $r_n = 0$ established the convergence result (24) for $t \in (0, 1)$ under a stronger Condition 1.

It was also shown in [8] that given Condition 2 with $|\psi| < 1$ and Condition 5 there exists a slowly varying sequence l_1, l_2, \dots such that

$$\mathcal{L}\left(n^{-\frac{1}{\alpha}} l_n \ln Z_{nt}, t \in [0, 1] \mid Z_n > 0\right) \xrightarrow{\mathcal{D}} \mathcal{L}(L_t^+, t \in [0, 1]) \text{ as } n \rightarrow \infty, \quad (25)$$

where $L_t^+, t \in [0, 1]$, denotes the meander of a strictly stable process with index α , which is a strictly stable Lévy process conditioned to stay positive on the time interval $(0, 1]$ (see [22] and [24]). Convergence (25) was earlier established by Kozlov [43] under the second moment assumption on the increments of the ARW.

Afanasyev [5] studied the behavior of the process up to the moment of extinction and proved that for a critical BPRE satisfying Condition 1 and some additional assumptions, as $n \rightarrow \infty$,

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} \ln(Z_{nt} + 1), t \in [0, \infty) \mid Z_n > 0\right) \xrightarrow{\mathcal{D}} \mathcal{L}(\mathbb{W}_t^+, t \in [0, \infty)).$$

The limiting process $\mathbb{W}_t^+, t \in [0, \infty)$, is a Brownian meander W_t^+ for $t \leq 1$ continued by a "stopped" Brownian motion $W_t, t \geq 1$ with $W_1 = W_1^+$ killed at the moment

$$\tau_0 = \inf\{t \geq 1 : W_t = 0\}.$$

Böinghoff, Dyakonova, Kersting, and Vatutin [20] consider a linear-fractional BPRE satisfying Conditions 1, 6, and Condition 5 with $\alpha = 2$. They have shown that

$$\mathcal{L}(Z_n \mid T = n + 1) \xrightarrow{d} \mathcal{L}(Y), \quad n \rightarrow \infty, \quad (26)$$

where Y is a positive integer-valued random variable being finite with probability 1, and furthermore, that for any $\delta \in (0, 1/2)$ as $n \rightarrow \infty$

$$\mathcal{L}\left(Z_{nte}^{-S_{nt}}, t \in [\delta, 1 - \delta] \mid T = n + 1\right) \xrightarrow{\mathcal{D}} \mathcal{L}(W_t, t \in [\delta, 1 - \delta]).$$

Here the limiting process is a stochastic process with a.s. constant paths, i.e., $\mathbb{P}(W_t = W \text{ for all } t \in [0, 1]) = 1$ for some positive random variable W , and convergence in distribution holds with respect to the Skorokhod topology in the space $D[\delta, 1 - \delta]$ of cadlag functions on the interval $[\delta, 1 - \delta]$.

Vatutin and Wachtel [64] have established for the critical BPREs whose increments of the ARWs belong to α -stable law with $\alpha < 2$ the following phenomenon of sudden extinction. Given the process dies at a remote time $T = n + 1$, the log-size $\ln Z_{nt}$ grows roughly as $n^{1/\alpha}$ up to moment n and then the process instantly dies out, so that $\ln Z_n$ is of order $n^{1/\alpha}$ while $Z_{n+1} = 0$. This may be interpreted as the evolution of the process in a favorable environment up to moment n stopped by a sudden extinction of the population at moment $T = n + 1$ due to a "catastrophic" change of the environment. Note that this phenomenon is in a sharp contrast with the case of finite variance (see

(26)) where the distribution of the number of individuals in the population just before the moment of its extinction converges to a distribution concentrated on the set of positive integers.

Let us formulate the results of [64] precisely. According to Durrett [24] under the Condition 2 there exists a nonnegative random variable Λ such that for a_n from (11)

$$\lim_{n \rightarrow \infty} \mathbb{P}(S_n \leq xa_n | S_1 > 0, \dots, S_n > 0) = \mathbb{P}(\Lambda \leq x) \quad \text{for all } x \geq 0.$$

It is proved in [64] that if Condition 2 holds restricted to $\alpha < 2$ and $|\psi| < 1$ together with Condition 5, then for every $x > 0$

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(Z_{n-1} > e^{xa_n}; T = n)}{\mathbb{P}(T^- = n)} = \Theta \frac{\mathbb{E}[\Lambda^{-\alpha}; \Lambda > x]}{\mathbb{E}[\Lambda^{-\alpha}]},$$

where (compare with (20))

$$\Theta := \liminf_{n \rightarrow \infty} \frac{\mathbb{P}(T = n)}{\mathbb{P}(T^- = n)}.$$

Moreover, it is found that in the geometric case

$$\lim_{n \rightarrow \infty} \mathbb{P}(Z_{n-1} > e^{xa_n} | T = n) = \frac{\mathbb{E}[\Lambda^{-\alpha}; \Lambda > x]}{\mathbb{E}[\Lambda^{-\alpha}]}.$$

3.2.2 Subcritical case

Weakly subcritical case. Afanasyev, Böinghoff, Kersting, and Vatutin [11] proved for weakly subcritical processes that under Conditions 3 and 7 the conditional laws $\mathcal{L}(Z_n | Z_n > 0)$, $n \geq 1$, converge weakly to some probability distribution on the natural numbers. Moreover, it was shown that the sequence $\mathbb{E}[Z_n^\vartheta | Z_n > 0]$ is bounded for any $\vartheta < \beta$, implying convergence to the corresponding moments of the limit distribution. It was also shown that for $X_t^{r,n}$ defined in (23)

$$\mathcal{L}(X_t^{r_n,n}, t \in [0, 1] | Z_n > 0) \xrightarrow{\mathcal{D}} \mathcal{L}(W_t, t \in [0, 1]), \quad n \rightarrow \infty,$$

weakly in the Skorokhod space $D[0, 1]$, where r_1, r_2, \dots are natural numbers such that $r_n < n/2$ and $r_n \rightarrow \infty$. Here $W_t = W$ a.s. for all $t \in [0, 1]$ and $\mathbb{P}(0 < W < \infty) = 1$. Compared to the critical case result (24) now both the initial and the most recent part of the population history are excluded. Earlier versions of this results can be found in [4] and [36].

Thus, we have the following pattern characterising the weakly subcritical case: given $Z_n > 0$ the value of Z_k are of bounded order for k close to 0 and close to n . For $1 \ll k \ll n$ the demographic dynamics of Z_k follows the value of the quenched conditional mean $\mathcal{E}[Z_k] = e^{S_k}$ in a completely deterministic manner up to a random factor $W > 0$, resembling the behavior of supercritical branching processes.

Intermediately subcritical case. For intermediately subcritical BPRES under the annealed approach Afanasyev, Böinghoff, Kersting, and Vatutin [10] proved that under Conditions 3 and 7 the unscaled population size Z_n conditioned on $\{Z_n > 0\}$ converges in distribution as $n \rightarrow \infty$. Under slightly different assumption this result was first obtained by Vatutin in [54].

Denote by \hat{L}_t the strictly stable process L_t with parameter α conditioned to have its minimum at time 1. Let e_1, e_2, \dots denote the excursion intervals of \hat{L}_t between consecutive local minima and put $j(t) = i$ for $t \in e_i$. In [10] it was also proved that Conditions 3 and 7 imply for $0 < t_1 < t_2 < \dots < t_k < 1$, as $n \rightarrow \infty$,

$$\mathcal{L} \left(\frac{Z_{nt_1}}{\exp(S_{nt_1} - \min_{k \leq nt_1} S_k)}, \dots, \frac{Z_{nt_k}}{\exp(S_{nt_k} - \min_{k \leq nt_k} S_k)} \mid Z_n > 0 \right) \xrightarrow{d} \mathcal{L}(V_{j(t_1)}, \dots, V_{j(t_k)}),$$

where V_1, V_2, \dots are i.i.d. copies of some strictly positive random variable V . This means that if t_i and t_k belong to one and the same excursion, then $j(t_i)$ and $j(t_k)$ coincide, i.e. $V_{j(t_i)} = V_{j(t_k)}$ with probability 1. Notice that here $j(t_1) \leq j(t_2) \leq \dots \leq j(t_n)$. For the linear fractional case this result was earlier obtained by Afanasyev in [7].

Recall that $\tau(nt)$ is the time of the first minimum of the ARW up to time $[nt]$. Complementing the previous results it was shown in [10] that at the times of consecutive ARW minima the population sizes have discrete limit distributions

$$\mathcal{L}(Z_{\tau(nt_1)}, \dots, Z_{\tau(nt_k)}) \mid Z_n > 0 \xrightarrow{d} \mathcal{L}(Y_{j(t_1)}, \dots, Y_{j(t_k)}),$$

where Y_1, Y_2, \dots are i.i.d. copies of a random variable Y taking values in \mathbf{N} .

Strongly subcritical case. Geiger, Kersting, and Vatutin [36] studied the strongly subcritical BPRES under condition (21) and proved that

$$\lim_{n \rightarrow \infty} \mathbb{P}(Z_n = z \mid Z_n > 0) = r_z, \quad z \in \mathbf{N},$$

where

$$\sum_{z=1}^{\infty} r_z = 1 \text{ and } m_r := \sum_{z=1}^{\infty} z r_z < \infty.$$

Afanasyev, Geiger, Kersting, and Vatutin [9] have introduced the following condition in the strongly subcritical case.

Condition 8 Suppose that (see (17))

$$\mathbb{E}[e^X \ln^+ \eta] < \infty.$$

Since in the strongly subcritical case $\mathbb{E}[e^X] < \infty$, Condition 8 holds in particular if the random offspring distribution (14) has uniformly bounded support. It also holds if (14) is a Poisson distribution with random mean, so that $\eta = 1$ a.s., or if (14) is a geometric distribution on \mathbf{N}_0 where $\eta = 2$ a.s.

It was shown in [9] that with

$$0 =: i_{n,0} < i_{n,1} < i_{n,2} < \dots < i_{n,k} < i_{n,k+1} := n$$

the following weak convergence holds

$$\mathcal{L}((Z_j)_{0 \leq j \leq m}, (Z_{i_{n,1}+j})_{0 \leq j \leq m}, \dots, (Z_{i_{n,k}+j})_{0 \leq j \leq m}, (Z_{n-j})_{0 \leq j \leq m} \mid Z_n > 0)$$

$$\xrightarrow{d} \mathcal{L}_{\delta_1}((Y_j)_{0 \leq j \leq m}) \otimes \mathcal{L}_{\hat{r}}((Y_j)_{0 \leq j \leq m})^{\otimes k} \otimes \mathcal{L}_r((\tilde{Y}_j)_{0 \leq j \leq m}),$$

for every $k, m \in \mathbf{N}_0$ as $\min_{0 \leq l \leq k} (i_{n,l+1} - i_{n,l}) \rightarrow \infty$ and $n \rightarrow \infty$. Here, $\mathcal{L}_\nu((Y_j)_{j \geq 0})$ denotes the law of the Markov chain $(Y_j)_{j \geq 0}$ with initial distribution ν and transitional probabilities

$$\hat{P}_{yz} = \frac{z\mathbb{P}(\xi_1 + \dots + \xi_y = z)}{y\mathbb{E}[\xi]}, \quad y, z \in \mathbf{N},$$

where ξ_1, \dots, ξ_y are independent copies of the offspring number ξ . The stationary distribution of $(Y_j)_{j \geq 0}$ is

$$\hat{r}_z = \frac{zr_z}{m_r}, \quad z \in \mathbf{N}.$$

The process $(\tilde{Y}_j)_{j \geq 0}$ is the time-reversed Markov chain with transition probabilities

$$\tilde{P}_{yz} = \frac{\hat{r}_z \hat{P}_{zy}}{\hat{r}_y}.$$

4 Multi-type BPRES

The model of multi-type BPRES was first considered by Athreya and Karlin [13] and subsequently investigated by Weissener [67], Kaplan [40], Tanny [49] (see also [66] for other references). Next we describe the BPRESs with p types of particles $\mathbf{Z}_n = (Z_{n,1}, \dots, Z_{n,p})$ and time $n = 0, 1, \dots$ using standard notation for the p -dimensional vectors:

- the unit vector \mathbf{e}_j has j -th component equals 1 and all others equal zero, $j = 1, \dots, p$,
- all zero and all one vectors $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$,
- for $\mathbf{x} = (x_1, \dots, x_p)$ and $\mathbf{y} = (y_1, \dots, y_p)$ set

$$\|\mathbf{x}\| = \sum_{i=1}^p |x_i|, \quad (\mathbf{x}, \mathbf{y}) = \sum_{i=1}^p x_i y_i, \quad \mathbf{x}^{\mathbf{y}} = \prod_{i=1}^p x_i^{y_i}.$$

A particle of type i in a multi-type branching process can produce z_1 particles of type 1, z_2 particles of type 2, ..., and z_p particles of type p with a probability $Q_{\mathbf{z}}^{(i)}$, $\mathbf{z} = (z_1, \dots, z_p)$. Under the random environment assumption the probabilities $Q_{\mathbf{z}}^{(i)}$ are random variables making random the corresponding PGFs

$$F^{(i)}(s_1, \dots, s_p) = \sum_{z_1=0}^{\infty} \dots \sum_{z_p=0}^{\infty} Q_{\mathbf{z}}^{(i)} \mathbf{s}^{\mathbf{z}}, \quad i = 1, \dots, p. \quad (27)$$

The random PGFs for the consecutive generations $(F_n^{(1)}, \dots, F_n^{(p)})_{n \geq 1}$ representing the environmental history determine the conditional mean offspring numbers as random $p \times p$ matrices \mathbf{M}_n with elements $\partial F_n^{(i)}(\mathbf{1}) / \partial s_j$. These are IID matrices having the same distribution as

$$\mathbf{M} = (M_{ij})_{i,j=1}^p, \quad M_{ij} = \frac{\partial F^{(i)}(\mathbf{1})}{\partial s_j}. \quad (28)$$

We will assume that elements M_{ij} are all positive meaning that each of p types of particles can produce any other type of particles in the next generation. We denote by R and R_n the Perron roots, i.e. the maximal (in absolute value) eigenvalues, for the matrices \mathbf{M} and \mathbf{M}_n . There exist left and right eigenvectors corresponding to the Perron eigenvalue

$$\begin{aligned} \mathbf{V} &= (V_1, \dots, V_p), \quad \mathbf{V}\mathbf{M} = R\mathbf{V}, \\ \mathbf{U} &= (U_1, \dots, U_p), \quad \mathbf{M}\mathbf{U}' = R\mathbf{U}', \end{aligned}$$

both with positive components and unique up to scaling. Although vectors \mathbf{V} and \mathbf{U} are random in general, throughout this section we assume that one of the following basic conditions is valid.

Condition 9 *There is a non-random strictly positive vector \mathbf{v} such that $\|\mathbf{v}\| = 1$ and*

$$\mathbb{P}(\mathbf{V} = \mathbf{v}) = 1.$$

Condition 10 *There is a non-random strictly positive vector \mathbf{u} such that $\|\mathbf{u}\| = 1$ and*

$$\mathbb{P}(\mathbf{U} = \mathbf{u}) = 1.$$

The important gain of this restriction is that we get either

$$\mathbf{v}\mathbf{M}_1 \dots \mathbf{M}_n = R_1 \dots R_n \mathbf{v}$$

or

$$\mathbf{M}_1 \dots \mathbf{M}_n \mathbf{u}' = R_1 \dots R_n \mathbf{u}'.$$

In both cases putting $X := \ln R$, $X_n := \ln R_n$, $n \geq 1$, we can again introduce an ARW by $S_n = X_1 + \dots + X_n$. We will refer to the conditions on S_n stated for the single type case even in the multi-type setting (under Conditions 9 or 10).

In particular, the single type classification for BPRES extends straightforwardly to this multi-type case. For example under Condition 9 as $n \rightarrow \infty$

$$\begin{aligned} \sum_{i=1}^p v_i \mathcal{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i) &\leq \min_{1 \leq k \leq n} |\mathbf{v} \mathbf{M}_1 \cdots \mathbf{M}_k| = \min_{1 \leq k \leq n} R_1 \cdots R_k \\ &= \exp\left\{\min_{1 \leq k \leq n} S_k\right\} \rightarrow 0 \end{aligned}$$

almost surely. This implies that in the critical and subcritical cases

$$\mathcal{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i) \rightarrow 0$$

almost surely for any $i = 1, \dots, p$. The same conclusion follows from Condition 10 as well.

4.1 The survival probability of the multi-type BPRES

Dyakonova and Vatutin have managed to extend some of the single type annealed results to the multi-type BPRES in the critical and subcritical cases. Some of the results mentioned in this section bring a stronger version of the known single type results.

4.1.1 Critical multi-type case

Dyakonova [28] considered a multi-type critical BPRES satisfying (3), Condition 9, and Condition 4, where $\zeta(a)$ is now defined as

$$\zeta(a) := R^{-2} \sum_{\mathbf{z} \in \mathbf{N}_a^p} \sum_{i=1}^p v_i \sum_{j,k=1}^p Q_{\mathbf{z}}^{(i)} z_j z_k, \quad (29)$$

with \mathbf{N}_a^p being the set of vectors \mathbf{z} with non-negative integer-valued components such that at least one component is larger or equal a . Here R is the (random) Perron root of the random matrix \mathbf{M} related to $Q_{\mathbf{z}}^{(i)}$ through (27) and (28). Additionally the following restriction (due to [41]) is imposed.

Condition 11 *There exists a number $0 < d < 1$ such that*

$$d \leq \frac{M_{i_1, j_1}}{M_{i_2, j_2}} \leq d^{-1}, 1 \leq i_1, i_2, j_1, j_2 \leq p.$$

Under these conditions it was proved in [28] that (compare with (18))

$$\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i) \sim c_i n^{-(1-\rho)} l_n, \quad i = 1, \dots, p, \quad (30)$$

where c_i are positive constants, ρ is the constant from the Spitzer condition applied to $S_n = \ln R_1 + \dots + \ln R_n$, and l_n is a sequence slowly varying at

infinity. Earlier the same asymptotics (30) was established in [26] in the linear fractional case under Condition 1 with $|\psi| < 1$ and Condition 10.

In [28] it was proved also a quenched type result claiming the almost sure convergence

$$\frac{\mathcal{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_i)}{\sum_{j=1}^p v_j \mathcal{P}(\mathbf{Z}_n \neq \mathbf{0} \mid \mathbf{Z}_0 = \mathbf{e}_j)} \rightarrow \mathbf{U}^*, \quad i = 1, \dots, p, \quad (31)$$

where $\mathbf{U}^* = (U_1^*, \dots, U_p^*)$ is a random vector such that

$$(\mathbf{v}, \mathbf{U}^*) = 1, \quad d \leq U_i^* \leq 1/\min(v_1, \dots, v_p), \quad i = 1, \dots, p.$$

Furthermore, it was shown in [28] that the above mentioned results remain true for an alternative set of conditions involving the counterparts of the single type Conditions 2 and 5.

Vatutin and Dyakonova [62] studied a multi-type critical BPFE whose ARW satisfies the Spitzer-Doney condition and the mean matrices of the reproduction laws have a common positive right eigenvector (see Condition 10). Also in [62] it is assumed that the following assumption holds.

Condition 12 *By this condition we exclude a possibility for the defining PGFs to take the linear form*

$$F^{(i)}(\mathbf{s}) = Q_{\mathbf{0}}^{(i)} + Q_{\mathbf{e}_1}^{(i)} s_1 + \dots + Q_{\mathbf{e}_p}^{(i)} s_p.$$

Another condition required in [62] concerns the moments for the vectors $(\xi_{i1}, \dots, \xi_{ip})$ of the offspring numbers having quenched distributions $\{Q_{\mathbf{z}}^{(i)}\}$ and quenched means (M_{i1}, \dots, M_{ip}) . For a given $\beta > 0$ set

$$\Delta_{ij}(\beta) = \mathcal{E} |\xi_{ij} - M_{ij}|^\beta, \quad \Delta_\beta = \max_{i,j} \Delta_{ij}(\beta)$$

and denote $\zeta_\beta := e^{-\beta X} \Delta_\beta$.

Condition 13 *There exist $\beta \in (1, 2]$ and $\varepsilon > 0$ such that*

$$\mathbb{P}(\Delta_\beta < \infty) = 1, \quad (32)$$

and

$$\mathbb{E}[\ln^+ \zeta_\beta]^{1/\rho+\varepsilon} < \infty, \quad \mathbb{E}[V(X)(\ln^+ \zeta_\beta)^{1+\varepsilon}] < \infty.$$

In [62] the asymptotic result (30) was established under (3) and Conditions 10, 12, and 13. The results of [62] generalize and extend not only the above mentioned results in [28] but also the corresponding single type statements from [8]. Indeed, in view of condition (32) the asymptotic relation (30) may be valid for $p = 1$ if, for instance,

$$\mathbb{P}(\mathcal{E} [\xi^2 1_{\{\xi \geq a\}}] = \infty) = 1$$

for any $a \geq 0$ making in this case the conditions of the last result weaker than those required in [8] for (18).

As in the single type case, denote by T the extinction moment for the process \mathbf{Z}_n . It was shown by Dyakonova [27] that in the multi-type linear fractional case under Conditions 1, 9, 10, 11 and some extra restrictions there exists a constant $0 < c < \infty$ such that (compare with (19)) for any $i = 1, \dots, p$

$$\mathbb{P}(T = n | \mathbf{Z}_0 = \mathbf{e}_i) \sim cu_i n^{-3/2}, \quad n \rightarrow \infty,$$

where $\mathbf{u} = (u_1, \dots, u_p)$ is defined in Condition 10.

4.1.2 Subcritical multi-type case

Dyakonova [29] investigated the asymptotics of the survival probability of multi-type subcritical BPRES. She has proved, that if Conditions 9 and 11 are valid and $\mathbb{E}[X] < 0$, then (31) holds \mathbb{P} -a.s., as $n \rightarrow \infty$. Furthermore, in [29] it was shown that in the strongly subcritical case as $n \rightarrow \infty$ (compare with (22))

$$\mathbb{P}(\mathbf{Z}_n \neq \mathbf{0} | \mathbf{Z}_0 = \mathbf{e}_i) \sim c_i (\mathbb{E}[R])^n, \quad c_i > 0, \quad i = 1, \dots, p$$

under Conditions 9, 11, and 8, with $\eta = \zeta(0)$ defined by (29).

4.2 Functional limit theorem in the critical case

Vatutin and Dyakonova [62] have assumed that assumption (3) is valid together with Conditions 10, 12, and Condition 13 with $\beta = 2$. They considered a family of the processes

$$W_{r,n}(t) := e^{-S_{r+(n-r)t}}(\mathbf{Z}_{r+(n-r)t}, \mathbf{u}), \quad t \in [0, 1]$$

for any given pair of integers $0 \leq r \leq n$. Letting r_1, r_2, \dots be a sequence of positive integers such that $r_n \leq n$ and $r_n \rightarrow \infty$, it was proved that as $n \rightarrow \infty$,

$$\mathcal{L}(W_{r_n,n}(t), t \in [0, 1] | \mathbf{Z}_n \neq \mathbf{0}; \mathbf{Z}_0 = \mathbf{z}) \xrightarrow{\mathcal{D}} \mathcal{L}(W_{\mathbf{z}}(t), t \in [0, 1]), \quad (33)$$

where the limiting process is a stochastic process with a.s. constant paths, i.e., $\mathbb{P}(W_{\mathbf{z}}(t) = W_{\mathbf{z}} \text{ for all } t \in [0, 1]) = 1$ with $\mathbb{P}(0 < W_{\mathbf{z}} < \infty) = 1$. Convergence (33) is a generalization of (24) in two directions. First, it is proved for the multitype case, and second, even for the single-type case the conditions under which (33) is established in [62] are weaker than those used in [8].

5 Quenched results for the critical BPRES

Vatutin and Dyakonova [56] and [57] applied the quenched approach to the critical BPRES meeting the Doney-Spitzer condition (3). Their results generalize those in [55] and [58] in which much stronger conditions on the characteristics of the branching processes are imposed.

5.1 Survival probability conditioned on the environment

The following condition involving the renewal functions (6) is dual to Condition 4 dealing with the renewal function (7).

Condition 14 *There are numbers $\varepsilon > 0$ and $a \in \mathbf{N}_0$ such that*

$$\mathbb{E}[(\ln^+ \zeta(a))^{\frac{1}{1-\rho}+\varepsilon}] < \infty \quad \text{and} \quad \mathbb{E}[U(-X)(\ln^+ \zeta(a))^{1+\varepsilon}] < \infty,$$

where $\zeta(a)$ is from (15).

It was proved in [57] that under (3) and Conditions 4, 14

$$e^{-S_{\tau(n)}} \mathcal{P}(Z_n > 0) \xrightarrow{d} \zeta, \quad n \rightarrow \infty, \quad (34)$$

where the random variable $\zeta \in [0, 1]$ is positive with probability 1. According to (34) the asymptotic behavior of the survival probability is mainly determined by the minimal value of the ARW over the time interval $[0, n]$.

5.2 Convergence of one-dimensional distributions

Denote

$$\hat{Y}_n := \frac{Z_n}{\mathcal{E}[Z_n | Z_n > 0]}$$

and

$$\mathcal{M}^{(n)}(dx) := \mathcal{P}(\hat{Y}_n \in dx | Z_n > 0).$$

Theorem 1 and Lemma 7 from [57] yield that under (3) and Conditions 4, 14 there exists a random measure \mathcal{M} (being proper and nondegenerate with probability 1) such that, as $n \rightarrow \infty$,

$$\int_0^\infty e^{-\lambda x} \mathcal{M}^{(n)}(dx) \xrightarrow{d} \Phi(\lambda) := \int_0^\infty e^{-\lambda x} \mathcal{M}(dx), \quad (35)$$

$$\int_0^\infty e^{-\lambda x} x \mathcal{M}^{(n)}(dx) \xrightarrow{d} \Psi(\lambda) := \int_0^\infty e^{-\lambda x} x \mathcal{M}(dx) = -\frac{d}{d\lambda} \Phi(\lambda). \quad (36)$$

In the linear-fractional case the limiting Laplace transform is deterministic $\Phi(\lambda) = (1 + \lambda)^{-1}$ and represents an exponential distribution with parameter 1. Notice that in this case $\Psi(\lambda) = (1 + \lambda)^{-2}$ corresponds to the sum of two independent $\text{Exp}(1)$ random variables. Thus in the linear-fractional case we get an analogue of the corresponding Yaglom-type limit theorem for the ordinary critical Galton-Watson processes.

For the case when the distribution of X is absolutely continuous convergences (34) and (35) were proved in [58].

5.3 Convergence of finite-dimensional distributions

For an integer $b \geq 2$ and tuples $\bar{t} = (t_0, t_1, \dots, t_b)$, $\bar{\lambda} = (\lambda_1, \dots, \lambda_b)$, where $0 = t_0 < t_1 < \dots < t_b = 1$ and $\lambda_i \geq 0$, $i = 1, \dots, b$, set

$$\Phi^{(n)}(\bar{t}, \bar{\lambda}) := \mathcal{E} \left[\exp \left\{ - \sum_{i=1}^b \lambda_i \widehat{Y}_{nt_i} \right\} \mid Z_n > 0 \right]$$

and for a positive integer $d \leq b$ and a vector $\bar{r} = (r_0, r_1, r_2, \dots, r_d)$ with integer-valued coordinates $0 = r_0 < r_1 < r_2 < \dots < r_d = b$ introduce the event

$$\begin{aligned} \mathcal{U}(\bar{t}; \bar{r}, n) &:= \left\{ \tau(nt_1) = \dots = \tau(nt_{r_1}) < \tau(nt_{r_1+1}) = \dots \right. \\ &\quad \left. = \tau(nt_{r_2}) < \dots < \tau(nt_{r_{d-1}+1}) = \dots = \tau(nt_{r_d}) \right\}. \end{aligned}$$

It was proved in [57] that if X meets Conditions 2, 4, 14, then, as $n \rightarrow \infty$,

$$\left\{ \Phi^{(n)}(\bar{t}, \bar{\lambda}) \mid \mathcal{U}(\bar{t}; \bar{r}, n) \right\} \xrightarrow{d} \prod_{i=1}^{d-1} \Psi_i \left(\sum_{j=r_{i-1}+1}^{r_i} \lambda_j \right) \Phi_d \left(\sum_{j=r_{d-1}+1}^b \lambda_j \right),$$

where $\Psi_i(\lambda)$, $i = 1, \dots, d-1$ and $\Phi_d(\lambda)$ are independent random functions distributed as $\Psi(\lambda)$ in (36) and $\Phi(\lambda)$ in (35) respectively.

Loosely speaking, this result shows that the trajectory (Z_0, \dots, Z_n) is partitioned into independent pieces generated by strict descending ladder moments of the ARW. The population size at the moment j is proportional to $e^{S_j - S_{\tau(j)}}$ while the maximal size of the population on the interval $[0, n]$ is proportional to $e^{\max_{0 \leq j \leq n} (S_j - S_{\tau(j)})}$ (with a random positive factor ≤ 1).

Let L_t be an α -stable Lévy process. For fixed $0 = t_0 < t_1 < \dots < t_b = 1$ put $\omega_p := \inf_{t_0 \leq u \leq t_p} L_u$, $0 \leq p \leq b$. Let D be the random number of different elements in the tuple $\omega_0, \omega_1, \dots, \omega_b$, and let \bar{R} be the random D -dimensional vector with components

$$R_0 = 0, R_{i+1} = \max \{ k \geq R_i + 1 : \omega_{R_{i+1}} = \omega_k \}$$

so that $R_D = b$. In [57] it was established that if X satisfies Conditions 2, 4, 14, then as $n \rightarrow \infty$

$$\Phi^{(n)}(\bar{t}, \bar{\lambda}) \xrightarrow{d} \prod_{i=1}^{D-1} \Psi_i \left(\sum_{j=R_{i-1}+1}^{R_i} \lambda_j \right) \Phi_D \left(\sum_{j=R_{D-1}+1}^b \lambda_j \right),$$

where $(\Psi_1(\lambda), \Phi_1(\lambda), \Psi_2(\lambda), \Phi_2(\lambda), \dots)$ are as before and independent of the Lévy process L_t defining the vector \bar{R} .

Relations (34) and (35) imply that if $t \in (0, 1]$ is *fixed* then, given $Z_n > 0$, the distribution of the random variable $Z_{nt} e^{S_{\tau(nt)} - S_{nt}}$ converges in the specified sense to a proper distribution with no atom at zero. This means, roughly speaking, that if the process survives up to moment n , then an earlier population

size Z_{nt} is proportional to $e^{S_{nt}-S_{\tau(nt)}}$. Thus, contrary to the conditional limit theorems for the classical critical or supercritical branching processes (in which the scaling functions for the population size increase with time either linearly or exponentially), the scaling function for the population size in the critical BPRE is subject to large random oscillations.

This indicates that the corresponding population passes through a number of bottlenecks at the moments around the consecutive points of minima of the ARW. Vatutin and Dyakonova in [55], [56], [57], [58], and [59] have investigated this phenomena in detail under the quenched approach. They shown that the distribution of the number of individuals in the process at the moments close to $\tau(nt)$, $t \in (0, 1]$ conditioned on survival up to time n converges to a discrete distribution. Thus, in contrast to the *fixed* points of the form nt where the size of the population is exponentially large (see [57]), the size of the population at the (random) point of global minimum of the ARW becomes drastically small but then it grows again exponentially. This reminds the typical demographics of real biological populations which during their evolution have "favorable periods" (rapid growth of the population size) and "unfavorable periods" (quick extinction when only a few representatives of the population survive) followed by another period of rapid growth. Note a similarity between this picture and the intermediately subcritical case under the annealed approach (see Section 3.2.2).

5.4 Discrete limit distributions

Consider the population size near the time of the global minimum of the ARW. Under (3) and Conditions 4, 14 it was proved in [59] that for any $m \in \mathbb{Z}$ and $t \in (0, 1)$

$$\begin{aligned} \mathcal{E} \left[s^{Z_{\tau(n)+m}} \mid Z_n > 0 \right] &\xrightarrow{d} \varphi_m(s), \\ \mathcal{L} \left(\mathcal{E} \left[s^{Z_{\tau(nt)+m}} \mid Z_n > 0 \right] \mid \tau(n) > nt \right) &\xrightarrow{d} \mathcal{L}(\varphi_m^*(s)), \end{aligned}$$

where $\varphi_m(s)$ and $\varphi_m^*(s)$ can be obtained precisely. Observe, that the second limiting PGF does not depend on a particular value of $t \in (0, 1)$.

Next we give another two convergence results from [59]. The first of them characterizes the conditional distribution of the number of individuals in the population at two sequential moments located in a vicinity of the moment when the global minimum of the associated random walk is attained, while the second convergence describes the same distribution in a vicinity of the moment $\tau(nt)$ given $\tau(n) > nt$. So, in [59] proved that under (3) and Conditions 4, 14 for any $m \in \mathbb{Z}$ and any $t \in (0, 1)$, $s_1, s_2 \in [0, 1)$, as $n \rightarrow \infty$,

$$\begin{aligned} \mathcal{E} \left[s_1^{Z(\tau(n)+m)} s_2^{Z(\tau(n)+m+1)} \mid Z(n) > 0 \right] &\xrightarrow{d} \varphi_m(s_1, s_2), \\ \mathcal{L} \left(\mathcal{E} \left[s_1^{Z(\tau(nt)+m)} s_2^{Z(\tau(nt)+m+1)} \mid Z(n) > 0 \right] \mid \tau(n) > nt \right) &\xrightarrow{d} \mathcal{L}(\varphi_m^*(s_1, s_2)), \end{aligned}$$

where $\varphi_m(s_1, s_2)$ and $\varphi_m^*(s_1, s_2)$ can be obtained precisely.

5.5 Conditioning on the precise time of extinction

Critical BPRES conditioned on extinction at a given moment were investigated by Vatutin and Kyprianou [63] using the quenched approach. Let

$$\tau(l, n) := \min \left\{ k \in [l, n] : S_k = \min_{l \leq p \leq n} S_p \right\}$$

be the left-most point on $[l, n]$ at which the minimal value of the ARW on the interval $[l, n]$ is attained. Assuming $\{T = n\}$ the authors of [63] proved conditional limit theorems describing the asymptotic behavior, as $n \rightarrow \infty$, of the distribution of the population sizes at moments nt , $t \in (0, 1)$ and at moments close to $\tau(nt)$. It turned out that if X belongs to the domain of attraction of a stable law with parameter $\alpha \in (0, 2]$, then (contrary to the annealed approach [64]) under the quenched approach the phenomenon of sudden extinction does not occur.

Let

$$A_{m,n} := \frac{\mathcal{P}(Z_n > 0 | Z_0 = 1)}{\mathcal{P}(Z_n > 0 | Z_m = 1)} \mathcal{P}(Z_n = 0 | Z_m = 1) b_m,$$

where $b_m := \sum_{j=0}^{m-1} \eta_{j+1} e^{-S_j} / 2$, and η_j are from (17). It is shown in [63] that in the linear-fractional case given assumption (3) and Conditions 4, 14 are valid the following is true. For any $m \in \mathbb{Z}$, $t \in (0, 1)$, $s \in (0, 1]$, $\lambda \in (0, \infty)$, as $n \rightarrow \infty$

$$\mathcal{L}(\mathcal{E}[s^{Z_{\tau(nt)+m}} | T = n] | \tau(n) \geq nt) \xrightarrow{d} \mathcal{L}\left(s \left(\frac{1 - \Theta_m}{1 - \Theta_m s} \right)^2\right), \quad (37)$$

$$\mathcal{L}(\mathcal{E}[s^{Z_{\tau(nt,n)+m}} | T = n] | \tau(n) < nt) \xrightarrow{d} \mathcal{L}\left(s \left(\frac{1 - \theta_m}{1 - \theta_m s} \right)^2\right), \quad (38)$$

where $\Theta_m \in (0, 1)$, $\theta_m \in (0, 1)$ with probability 1, and furthermore

$$\mathcal{E}\left[\exp\left\{-\lambda \frac{Z_{nt}}{A_{nt,n}}\right\} | T = n\right] \xrightarrow{d} \frac{1}{(1+\lambda)^2}. \quad (39)$$

According to [63] on the set $\tau(n) \geq nt$ the random variable $A_{nt,n} e^{S_{\tau(nt)} - S_{nt}}$ converges in distribution, as $n \rightarrow \infty$, to a random variable being finite and positive with probability 1. Thus, for such moments nt the normalization in (39) is essentially specified by the past behavior of the ARW. On the other hand, given $\tau(n) < nt$ the random variable $A_{nt,n} e^{S_{\tau(nt,n)} - S_{nt}}$ converges in distribution, as $n \rightarrow \infty$, to a random variable being finite and positive with probability 1. Thus, for such moments nt the scaling in (39) is essentially specified by the future behavior of the ARW.

This fact allows us to give the following non-rigorous interpretation of the mentioned results. If the process dies out at a distant moment $T = n$, then it happens not as a unique catastrophic event. Before the extinction moment the evolution of the process consists of a number of "bad" periods characterized by

small population sizes. According to (37) and (38), such periods are located in the vicinities of *random* points $\tau(nt)1_{\{\tau(n) \geq nt\}}$ and $\tau(nt, n)1_{\{\tau(n) < nt\}}$. On the other hand, at *nonrandom* points nt , $t \in (0, 1)$, the size of the population by (39) is big. Hence $\ln Z_{nt}$ grows like $S_{nt} - S_{\tau(nt)}$ if $\tau(n) > nt$ and like $S_{nt} - S_{\tau(nt, n)}$ if $\tau(n) < nt$. Thus, the process $\{Z_{tT}, 0 \leq t \leq 1\}$ dies by passing through a number of bottlenecks and favorable periods.

6 Reduced BPRES

The next stage in studying branching processes is to investigate the structure of their genealogical trees. For $0 \leq k \leq m$ let $Z_{k,m}$ be the number of particles at moment k in the process $\{Z_n, n \geq 0\}$ each of which has a nonempty offspring at moment m . The tuple $\{Z_{k,m}, 0 \leq k \leq m < \infty\}$ is called the reduced branching process which gives the number of branches in the genealogical tree of the population modeled by the branching process $\{Z_n, n \geq 0\}$.

6.1 Annealed approach

The first results for reduced BPRES with linear-fractional PGFs and under the annealed approach were established by Borovkov and Vatutin [19] for the critical case and by Fleischmann and Vatutin [34] for all three types of subcritical BPRES. For the critical case Vatutin [52] proved under Condition 1 and some additional conditions that

$$\mathcal{L}\left(\frac{1}{\sigma\sqrt{n}} \ln Z_{nt,n}, t \in [0, 1] \mid Z_n > 0\right) \xrightarrow{\mathcal{D}} \mathcal{L}\left(\inf_{t \leq u \leq 1} W_u^+, t \in [0, 1]\right), \quad (40)$$

where W_t^+ , $t \in [0, \infty)$, is the Brownian meander and the convergence in distribution holds in Skorokhod topology in space $D[0, 1]$. This convergence was established in [19] for the case of linear-fractional BPRES under stronger moment assumptions.

Let $B_n = \max\{m < n : Z_{m,n} = 1\}$. The difference $d_n = n - B_n$ is called the time to the most recent common ancestor for all individuals existing at time n (MRCA_n). For ordinary Galton-Watson processes conditioned on survival, the classical results by Zubkov [68] state that in the critical case d_n is asymptotically uniformly distributed over $[0, n]$ while in the subcritical case d_n is asymptotically finite. We have a quite different situation in the random environment setting.

For the annealed approach convergence (40) states, roughly speaking, that the number of individuals $Z_{nt,n}$ grows as $\exp\{\sqrt{n} \inf_{t \leq u \leq 1} W_u^+\}$. Recalling that $\mathbf{P}(\inf_{t \leq u \leq 1} W_u^+ > 0) = 1$ for any $t \in (0, 1]$, we conclude that in the annealed setting conditionally on survival the MRCA_n has lived at early time of order $o(n)$. In fact, as shown by Borovkov and Vatutin [19] for the linear-fractional case, the MRCA_n with positive probability is the initial individual!

Studying the subcritical BPRES with geometric offspring distributions Fleischmann and Vatutin [34] have found that for the intermediately and strongly

subcritical processes the MRCA_n is located not too far from the point of observation n , while for the weakly subcritical case it may be located either nearby of the point of observation or at the very beginning of the evolution of the process.

6.2 Quenched results for the critical reduced BPRES

This section contains a summary of results established by Vatutin and Dyakonova in [60] and [61] for the critical reduced BPRES under the quenched approach. The results formulated in this section are proved under (3) and Conditions 4, 14, if not explicitly stated otherwise.

6.2.1 Time to the MRCA and afterwards

Vatutin and Dyakonova [60] proved that in the critical case the MRCA_n is located "not too far" from the moment of the global minimum $\tau(n)$ of the ARW on $[0, n]$ in that for any $m \in \mathbb{Z}$

$$\mathcal{P}(B_n = \tau(n) + m) \xrightarrow{d} r_m, \quad \sum_{m=-\infty}^{\infty} r_m = 1,$$

and with probability 1

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathcal{P}(|B_n - \tau(n)| \leq m) = 1.$$

Recalling (5) we clearly see the difference with the ordinary Galton-Watson critical branching processes where the time to the most recent common ancestor, scaled by n , is uniformly distributed on $[0, 1]$.

Also in [60] it was proven that the finite-dimensional quenched distributions of the process $\{Z_{\tau(n)+m, n}, m \in \mathbb{Z}\}$ conditioned on $Z_n > 0$ weakly converge to the finite-dimensional distributions of a Galton-Watson branching process evolving in an inhomogeneous random environment. The limit process starts at $-\infty$ by a single individual. The founder of the population dies at a moment $m \in (-\infty, +\infty)$ with probability r_m producing at least one offspring in accordance with the PGF $a_m(s)$ defined as the weak limit

$$\mathcal{E}[s^{Z_{\tau(n)+m+1, n}} | Z_{\tau(n)+m, n} = 1] \xrightarrow{d} a_m(s), \quad s \in [0, 1),$$

as $n \rightarrow \infty$. The next generation particles reproduce independently according to the PGF $a_{m+1}(s)$. And so on. Observe that this description is in a sharp contrast with the respective limit process for the ordinary critical reduced Galton-Watson processes obtained in [33].

The properties of reduced processes far to the right of $\tau(n)$ were studied in [60] as well. Using the scaling function

$$\beta_n(k) := \frac{1}{\mathcal{E}[Z_{k, n} | Z_n > 0]}, \quad 0 \leq k \leq n,$$

it was shown that as $n \rightarrow \infty$ and $m \rightarrow \infty$ with $\tau(n) + m \leq n$

$$\mathcal{E} \left[e^{-\lambda Z_{\tau(n)+m, n} \beta_n(\tau(n)+m)} \mid Z_n > 0 \right] \xrightarrow{d} \Phi(\lambda),$$

where $\Phi(\lambda)$ is the same random Laplace transform as in (35). Moreover, if $0 \leq m_1 < m_2 < \dots < m_p \leq n - \tau(n)$, then as $m_1 \rightarrow \infty$ and $n \rightarrow \infty$

$$\mathcal{E} \left[\prod_{i=1}^p e^{-\lambda_i Z_{\tau(n)+m_i, n} \beta_n(\tau(n)+m_i)} \mid Z_n > 0 \right] \xrightarrow{d} \Phi \left(\sum_{i=1}^p \lambda_i \right).$$

6.2.2 Reduced processes at nonrandom times

Properties of the reduced process $Z_{nt, n}$ for *nonrandom* $t \in (0, 1)$ were also described in [60]. It was shown that

$$\mathcal{E} \left[e^{-\lambda Z_{nt, n} \beta_n(nt)} \mid Z_n > 0 \right] \xrightarrow{d} \Phi(\lambda) 1_{\{\tau \leq t\}} + e^{-\lambda} 1_{\{\tau > t\}}, \quad n \rightarrow \infty,$$

where τ is a random point having a $\text{Beta}(1 - \rho, \rho)$ distribution as in (5) and being independent from $\Phi(\lambda)$. Furthermore, for any $\lambda_1 \geq 0, \dots, \lambda_k \geq 0$ and $0 < t_1 < \dots < t_k \leq 1$, as $n \rightarrow \infty$,

$$\mathcal{E} \left[\exp \left\{ - \sum_{i=1}^k \lambda_i Z_{nt_i, n} \beta_n(nt_i) \right\} \mid Z_n > 0 \right] \xrightarrow{d} \Phi \left(\sum_{i=i_\tau}^k \lambda_i \right) e^{-\sum_{i=1}^{i_\tau-1} \lambda_i},$$

where $i_\tau = \min\{i : \tau < t_i\}$.

The corresponding limiting process $\{W_t, 0 \leq t \leq 1\}$ follows a simple pattern (conditioned on a given environmental development). Until time τ (corresponding to the time to the MRCA) we have $W_t = 1$ and after time τ we have $W_t \equiv Y$, where the random value Y has $\Phi(\lambda)$ as its Laplace transform. Recall that in the linear-fractional case Y is exponentially distributed with parameter 1.

Concerning the scaling function $\beta_n(nt)$ it is known from [60] that under Conditions 2, 4, 14 for any $t \in (0, 1)$

$$\mathcal{L} \left(e^{S_{\tau(nt, n)} - S_{\tau(n)}} \beta_n(nt) \mid \tau(n) < nt \right) \xrightarrow{d} \mathcal{L}(\zeta^*), \quad n \rightarrow \infty,$$

where ζ^* is a proper random variable. It follows that if the global minimum time $\tau(n)$ of the ARW occurs prior to nt then $Z_{nt, n}$ is of order $e^{S_{\tau(nt, n)} - S_{\tau(n)}}$ (up to a random multiplier separated from zero and infinity).

6.2.3 Reduced BPRES as random fields

Vatutin and Dyakonova [61] obtained quenched limit theorems for the random fields $\{Z_{nt_1, nt_2}\}_{0 < t_1 < t_2 < 1}$ as $Z_n > 0$ and $n \rightarrow \infty$. The answers essentially

depend on the position of $\tau(n)$ relative to the interval $[nt_1, nt_2]$: as $n \rightarrow \infty$

$$\mathcal{L} \left(\mathcal{E}[e^{-\lambda Z_{nt_1, nt_2} \beta_n(t_1, t_2)} | Z_n > 0] \mid \tau(n) < nt_1 \right) \xrightarrow{d} \Phi(\lambda), \quad (41)$$

$$\mathcal{L} \left(\mathcal{E}[e^{-\lambda Z_{nt_1, nt_2} \beta_n(t_1, t_2)} | Z_n > 0] \mid \tau(n) > nt_2, \tau(nt_2) \leq nt_1 \right) \xrightarrow{d} \Psi(\lambda), \quad (42)$$

$$\mathcal{L} \left(\mathcal{E}[s^{Z_{nt_1, nt_2}} | Z_n > 0] \mid nt_1 \leq \tau(n) \leq nt_2 \right) \xrightarrow{d} s, \quad (43)$$

$$\mathcal{L} \left(\mathcal{E}[s^{Z_{nt_1, nt_2}} | Z_n > 0] \mid \tau(n) > nt_2, \tau(nt_2) > nt_1 \right) \xrightarrow{d} s, \quad (44)$$

where $\Phi(\lambda)$ is from (35), $\Psi(\lambda)$ is from (36), and

$$\beta_n(t_1, t_2) := \frac{1}{\mathcal{E}[Z_{nt_1, nt_2} | Z_{nt_2} > 0]} = \frac{1 - f_{0, nt_2}(0)}{e^{S_{nt_1}} (1 - f_{nt_1, nt_2}(0))}.$$

According to Theorem 1 in [56]

$$\beta_n(t_1, t_2) e^{S_{\tau(nt_1, nt_2)} - S_{\tau(nt_1)}} \xrightarrow{d} \hat{\Theta}, \quad n \rightarrow \infty,$$

where the random variable $\hat{\Theta}$, is positive and finite with probability 1. Observe that in the cases (43) and (44) we have $S_{\tau(nt_1, nt_2)} \leq S_{\tau(nt_1)}$. Results (43) and (44) say that in these cases all individuals existing at moment nt_2 are descendants from a single individual existing at moment nt_1 . Relations (41) and (42) describe the cases when the corresponding number of ancestors is large with $Z_{nt_1, nt_2} \asymp e^{S_{\tau(nt_1, nt_2)} - S_{\tau(nt_1)}}$.

References

- [1] V.I. Afanasyev, *Limit theorems for a conditional random walk and some applications*. Diss. Cand. Sci., Moscow, MSU, 1980.
- [2] V.I. Afanasyev, *Limit theorem for the critical branching process in random environment*. Discrete Math. Appl., 5 (1993), pp. 45–58.
- [3] V.I. Afanasyev, *A new theorem for a critical branching process in random environment*. Discrete Math. Appl., 7 (1997), pp. 497–514.
- [4] V.I. Afanasyev, *Limit theorems for a moderately subcritical branching process in a random environment*. Discrete Math. Appl., 8 (1998), pp. 55–72.
- [5] V.I. Afanasyev, *On the maximum of a critical branching process in a random environment*. Discrete Math. Appl., 9 (1999), pp. 267–284.
- [6] V.I. Afanasyev, *A functional limit theorem for a critical branching process in random environment*. Discrete Math. Appl., 11 (2001), pp. 587–606.
- [7] V.I. Afanasyev, *Limit theorems for an intermediately subcritical and a strongly subcritical branching process in a random environment*. Discrete Math. Appl., 11 (2001), pp. 105–131.

- [8] V.I. Afanasyev, J. Geiger, G. Kersting, and V.A. Vatutin, *Criticality for branching processes in random environment*. Ann. Prob., 33 (2005), pp. 645–673.
- [9] Afanasyev V.I., Geiger J., Kersting G., Vatutin V.A. *Functional limit theorems for strongly subcritical branching processes in random environment*. Stoch. Proc. Appl., 115 (2005), pp. 1658–1676.
- [10] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin, *Conditional limit theorems for intermediately subcritical branching processes in random environment*. (submitted), (2011). <http://arxiv.org/abs/1108.2127>
- [11] V. I. Afanasyev, C. Böinghoff, G. Kersting, and V. A. Vatutin, *Limit theorems for weakly subcritical branching processes in random environment*. to appear in J. Theoret. Probab., DOI: 10.1007/s10959-010-0331-6 (2010).
- [12] S. Albeverio, M.V. Kozlov, *On the recurrence and transience of state-dependent branching processes in random environment*. Theory Probab. Appl., 48 (2003), pp. 575–591.
- [13] K.B. Athreya, S. Karlin, *Branching processes with random environments, I: Extinction probabilities*. Ann. Math. Statist., 42 (1971), pp. 1499–1520.
- [14] K.B. Athreya, S. Karlin, *Branching processes with random environments, II: Limit theorems*. Ann. Math. Statist., 42 (1971), pp. 1843–1858.
- [15] K.B. Athreya, P.E. Ney, *Branching Processes*. Springer, Berlin, 1972.
- [16] P. Billingsley, *Convergence of probability measures*. Wiley, New York, 1968.
- [17] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*. Paperback ed., Cambridge University Press, Cambridge, 1989.
- [18] M. Birkner, J. Geiger, and G. Kersting, *Branching processes in random environment - a view on critical and subcritical cases*. In: J.-D. Deuschel and A. Greven, Editors, *Interacting Stochastic Systems*, Springer, Berlin, 2005, pp. 269–291.
- [19] K.A. Borovkov, V.A. Vatutin, *Reduced critical branching processes in random environment*. Stoch. Proc. Appl., 71 (1997), pp. 225–240.
- [20] C. Böinghoff, E.E. Dyakonova, G. Kersting, and V.A. Vatutin, *Branching processes in random environment which extinct at a given moment*. Markov Process. Relat. Fields, 16 (2010), 329–350.
- [21] F.M. Dekking, *On the survival probability of a branching process in a finite state i.i.d. environment*. Stoch. Proc. Appl., 27 (1988), pp. 151–157.
- [22] R.A. Doney, *Conditional limit theorems for asymptotically stable random walks*. Z. Wahrscheinlichkeitsth. verw. Geb., 70 (1985), pp. 351–360.

- [23] R.A. Doney, *Spitzer's condition and the ladder variables in random walks*. Probab. Theory Relat. Fields, 101 (1995), pp. 577–580.
- [24] R. Durrett, *Conditioned limit theorems for some null recurrent Markov processes*. Ann. Probab., 6 (1978), pp. 798–827.
- [25] R.T. Durrett, D.L. Iglehart and D.R. Miller, *Weak convergence to Brownian meander and Brownian excursion*. Ann. Probab., 5 (1977), pp. 117–129.
- [26] E.E. Dyakonova, *Asymptotic behavior of the probability of non-extinction for a multitype branching process in random environment*. Discrete Math. Appl., 9 (1999), pp. 119–136.
- [27] E.E. Dyakonova, *On multitype branching processes in a random environment*. J. Math. Sciences, 111 (2002), pp. 3537 – 3541.
- [28] E.E. Dyakonova, *Critical multitype branching process in random environment*. Discrete Math. Appl., 17 (2007), pp. 587– 606.
- [29] E.E. Dyakonova, *On subcritical multi-type branching processes in random environment*. Proceedings of the Fifth Colloquium on Mathematics and Computer Science. Algorithms, Trees, Combinatorics and Probabilities, Nancy: DMTCS, 2008, pp. 401–408.
- [30] E.E. Dyakonova, J. Geiger and V.A. Vatutin, *On the survival probability and a functional limit theorem for branching processes in random environment*. Markov Process. Related Fields, 10 (2004), pp. 289–306.
- [31] W. Feller, *An Introduction to Probability Theory and its Applications*. V.2, Wiley, New York - London-Sydney-Toronto, 1971.
- [32] K.Fleischmann, U.Prehn, *Ein Grenzfesatz für subkritische Verzweigungsprozesse mit endlich vielen Typen von Teilchen*. Math. Nachr., 64 (1974), pp. 233–241.
- [33] K. Fleischmann , R. Siegmund-Schultze, *The structure of reduced critical Galton–Watson processes*. Math. Nachr., 79 (1977), pp. 357–362.
- [34] K. Fleischmann, V.A. Vatutin, *Reduced subcritical branching processes in random environment*. Adv. Appl. Probab., 31 (1999), pp. 88–111.
- [35] J. Geiger and G. Kersting, *The survival probability of a critical branching process in random environment*. Theory Probab. Appl., 45 (2000), pp. 607–615.
- [36] J. Geiger, G. Kersting, and V.A. Vatutin, *Limit theorems for subcritical branching processes in random environment*. Ann. I.H. Poicare, 39 (2003), pp. 593–620.

- [37] Y. Guivarc'h, Q. Liu, *Propriétés asymptotiques des processus de branchement en environnement aléatoire*. C.R. Acad. Sci. Paris, Ser. I Math., 332 (2001), pp. 339–344.
- [38] P. Jagers, *Galton-Watson processes in varying environments*. J. Appl. Probab., 11 (1974), pp. 174–178.
- [39] M. Jirina, *Extinction of non-homogeneous Galton-Watson processes*. J. Appl. Probab., 13 (1976), pp. 132–137.
- [40] N. Kaplan, *Some results about multidimensional branching processes with random environments*. Ann. Probab., 2 (1974), pp. 441–455.
- [41] H. Kesten, F. Spitzer, *Convergence in distribution of products of random matrices*. Z. Wahrsch. verw. Geb., 67 (1984), pp. 363–386.
- [42] M.V. Kozlov, *On the asymptotic behavior of the probability of non-extinction for critical branching processes in a random environment*. Theory Probab. Appl., 21 (1976), pp. 791–804.
- [43] M.V. Kozlov, *A conditional function limit theorem for a critical branching process in a random medium*. Dokl. Akad. Nauk., 344 (1995), pp. 12–15 (in Russian).
- [44] Q. Liu, *On the survival probability of a branching process in a random environment*. Ann. Inst. Henri Poincaré Prob. et Statist., 32 (1996), pp. 1–10.
- [45] B.A. Sevastyanov, *Verzweigungsprozesse*. Akademie, Berlin, 1974.
- [46] F. Spitzer, *Principles of random walk*. Princeton NJ, Toronto - New York - London, 1964.
- [47] W.L. Smith, W.E. Wilkinson, *On branching processes in random environment*. Ann. Math. Statist., 40 (1969), pp. 814–827.
- [48] J.S.D'Souza, B.M. Hambly, *On the survival probability of a branching process in a random environment*. Adv. Appl. Probab., 29 (1997), pp. 38–55.
- [49] D. Tanny, *Limit theorems for branching processes in a random environment*. Ann. Probab., 5(1977), pp. 100–116.
- [50] D. Tanny, *On multitype branching processes in a random environment*. Adv. Appl. Probab., 13 (1981), pp. 464–497.
- [51] V.A. Vatutin, *Distance to the nearest mutual ancestor in the Bellman-Harris branching processes*. Math. Notes, 25 (1979), pp. 378–387.
- [52] V.A. Vatutin, *Reduced processes in random environment: the critical case*. Theory Probab. Appl., 47 (2002), pp. 99–113.

- [53] V.A. Vatutin, E.E. Dyakonova, *Critical branching processes in random environment: the probability of extinction at a given moment*. Discrete Math. Appl., 7 (1997), pp. 469–496 .
- [54] V.A. Vatutin, *A limit theorem for intermediate subcritical branching processes in a random environment*. Theory Probab. Appl., 48 (2003), pp. 481–492.
- [55] V.A. Vatutin , E.E. Dyakonova, *Reduced branching processes in random environment*. Mathematics and Computer Science II: Algorithms, Trees, Combinatorics and Probabilities (Ed. B.Chauvin, P.Flajolet, D.Gardy, A.Mokkadem), Birkhäuser, Basel - Boston- Berlin, 2002, pp. 455–467.
- [56] V.A. Vatutin, E.E. Dyakonova, *Galton-Watson branching processes in random environment, I: limit theorems*. Theory Probab. Appl., 48 (2003), pp. 314–336.
- [57] V.A. Vatutin, E.E. Dyakonova, *Galton-Watson branching processes in random environment, II: joint distributions*. Theory Probab. Appl., 49 (2004), pp. 275–308.
- [58] V.A. Vatutin, E.E. Dyakonova, *Yaglom type limit theorems for branching processes in random environment*. Mathematics and Computer Science III: Algorithms, Trees, Combinatorics and Probabilities (Ed. M.Drmota, P.Flajolet, D.Gardy, B.Gittenberger), Birkhäuser, Basel - Boston- Berlin, 2004, pp. 375–385.
- [59] V.A. Vatutin, E.E. Dyakonova, *Branching processes in random environment and bottlenecks in the evolution of populations*. Theory Probab. Appl., 51 (2006), pp. 189–210.
- [60] V.A. Vatutin, E.E. Dyakonova, *Limit theorems for reduced branching processes in random environment*. Theory Probab. Appl., 52 (2007), pp. 277–302.
- [61] V.A. Vatutin, E.E. Dyakonova, *Waves in reduced branching processes in random environment*. Theory Probab. Appl., 53 (2008), pp. 679–695.
- [62] V.A. Vatutin, E.E. Dyakonova, *Asymptotic properties of multitype critical branching processes evolving in random environment*. Discrete Math. Appl., 20 (2010), pp. 157–177.
- [63] V.A. Vatutin, A.E.Kyprianou, *Branching processes in random environment die slowly*. Fifth Colloquium on Mathematics and Computer Science, September 22-26, 2008, Blaubeuren, Germany. DMTCS Proceedings Series Volume AG, pp. 379–400 (electronic: <http://www.dmtcs-ojs/index.php/proceedings/issue/view/dmAG>).
- [64] V.A. Vatutin, V.Wachtel, *Sudden death of a critical branching process in random environment*. Theory Probab. Appl., 54 (2009), pp. 466–484.

- [65] Vatutin V.A., V. Wachtel, *Local probabilities for random walks conditioned to stay positive*. Probab. Theory Related Fields, 143 (2009), N 1-2, pp. 177–217.
- [66] V.A.Vatutin, A.M. Zubkov, *Branching Processes II*. J. Soviet Mathematics, 67 (1993), 3407–3485.
- [67] E. W. Weissener, *Multitype branching processes in random environments*. J. Appl. Prob., 8 (1971), pp. 17–31.
- [68] A.M. Zubkov, *Limit distributions of the distance to the nearest common ancestor*. Theory Probab. Appl., 20 (1975), pp. 602–612.